Problem 3.14

Prove any $a_n \in \mathbb{R}$ has a monotonic subsequence.

proof: (1) Assume $\exists\, a_{n_k}$ with no least term.

Let $k_1 = 1$, so $a_{n_{k_1}} = a_{n_1}$.

Since a_{n_k} does not have a least term,

$\exists\ k_2 > k_1$. s.t. $a_{n_{k_1}} > a_{n_{k_2}}$

Similarly. $\exists\ k_3 > k_2 > k_1$, s.t. $a_{n_{k_1}} > a_{n_{k_2}} > a_{n_{k_3}}$

So, a_{n_k} has a subsequence $a_{n_{k_l}}$ that is decreasing, and it is also a subsequence of a_n.

(2) Assume every a_{n_k} has a least term.

Let n_1 s.t. $a_{n_1} \leqslant a_n$, $\forall\, n \in \mathbb{N}$.

Let n_2 s.t. $a_{n_2} \leqslant a_n$, $\forall\, n > n_1$

Let n_3 s.t. $a_{n_3} \leqslant a_n$, $\forall\, n > n_2$.

As a result, $a_{n_1} \leqslant a_{n_2} \leqslant a_{n_3} \leqslant \cdots$ □

Problem 4.18

$f: E \mapsto E$, $f(x) = x$, $\forall x \in E$.

Prove f is uniformly continuous for any E.

proof: Let $\varepsilon > 0$, Let $\delta = \varepsilon$.

if $d(x, y) < \delta$, $d(f(x), f(y))$

$= d(x, y) < \delta = \varepsilon$.

Problem 4.16

$f: E \mapsto E'$ continuous, one-to-one onto

E compact $\Rightarrow f': E' \mapsto E$ continuous.

proof: Recall that f continuous $\Leftrightarrow$

$f^{-1}(U')$ is open for any U' open.

So, we want to show that

$(f^{-1})^{-1}(U)$ is open for any U open.

Since f is one-to-one open,

$$(f^{-1})^{-1}(U) = f(U)$$

Indeed, $f(U) = \{y \in E' : y = f(x) \text{ for some } x \in U\}$,

$$(f^{-1})^{-1}(U) = \{y \in E' : f^{-1}(y) \in U\}.$$

Let $y \in (f^{-1})^{-1}(U)$, $\quad f^{-1}(y) \in U$

$\Rightarrow \quad f(f^{-1}(y)) = y \in f(U)$.

Let $y \in f(U)$, for some $x \in U$

$\Rightarrow \quad f^{-1}(y) = f^{-1}(f(x)) = x \in U$

$\Rightarrow \quad y \in (f^{-1})^{-1}(U)$.

Let $U \subset E$ open, then $\bar{U}$ is closed.

Since $\begin{cases} \bar{U} \subset E \text{ closed} \\ E \text{ is compact} \end{cases}$

$\Rightarrow \quad \bar{U}$ is compact.

$\Rightarrow \quad f(\bar{U})$ is also compact.

$\Rightarrow \quad f(\bar{U})$ is closed in E

one-to-one onto

$$\text{Since } \begin{cases} f(U) \perp\!\!\!\perp f(\bar{U}) = E' \\ f(\bar{U}) \text{ is closed} \end{cases}$$

$\Rightarrow f(U)$ is open. $\square$

<u>Example</u>

(1) $$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

~~proof~~: Let $\varepsilon > 0$. $|x| < \delta$ and $|y| < \delta$.

$$\left| \frac{xy^2}{x^2+y^2} \right| = \left| \frac{1}{x} \cdot \frac{1}{1/x^2 + 1/y^2} \right|$$

$$\leq \left| \frac{1}{x} \cdot \sqrt{x^2 y^2} \right|$$

$$= |y| < \varepsilon \quad \text{if } \delta = \varepsilon.$$

(2) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Problem 4.21

$S \subset E$, $\bar{S} = \{ x \notin S : x \text{ is a cluster point of } S \}$.

$f : S \mapsto E'$, E' complete.

f is uniformly continuous.

Prove f can be extended to E and result in a continuous function.

~~proof~~: Let $c \in \overline{S}$, since c is a cluster point of S. $\forall n \in \mathbb{N}$, $\exists\, S_n \in S$ s.t. $d(c, S_n) < \frac{1}{n}$, then $S_n \to c$.

To show $f(x) = \lim_{n\to\infty} f(S_n)$, we need to show $f(S_n)$ is Cauchy since E' is complete.

Let $\varepsilon > 0$, since f is continuous. $\exists\, \delta > 0$, s.t.

$$d(S_n, S_m) < \delta \Rightarrow d(f(S_n), f(S_m)) < \varepsilon.$$

Let $N = \frac{2}{\delta}$, if $m, n > N$, then

$$d(S_m, S_n) \leq d(x, S_m) + d(x, S_n)$$

$$< \frac{1}{m} + \frac{1}{n} < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

$$\Rightarrow d(f(S_m), f(S_n))) < \varepsilon.$$

Thus, $f(S_n)$ is Cauchy. So, $\exists\, f(x) \in E'$ such that $\lim_{n\to\infty} f(S_n) = f(x)$. ($E'$ is complete)

$\Rightarrow f(x)$ can be extended to E.

Now, we want to show that the extended f' is uniformly continuous in E. We need to cover three cases below.

<u>Case 1</u>: $\bar{S} = \emptyset$. Then $f' = f$ and $E = S$. Since f is uniformly continuous in S, so does f'.

<u>Case 2</u>: $\bar{S}$ has only one element. Let $\bar{S} = \{c\}$. Let $s_n \to c$, as proved before, $f(s_n)$ is Cauchy. By construction, we know that $f(s_n) \to f'(c)$. So, f' is continuous at c.

So, let $\varepsilon > 0$. $\exists\ \delta_1 > 0$, s.t. $d(x, c) < \delta_1$ $\Rightarrow d(f'(x), f'(c)) < \varepsilon$.

On the other hand, since f is uniformly continuous in S. $\exists\ \delta_2 > 0$, s.t. $d(x, y) < \delta_2$, $x, y \in S$ $\Rightarrow d(f(x), f(y)) < \delta_2$.

Now, select $\delta = \min\{\delta_1, \delta_2\}$. $\forall\ x, y \in E$,

$\delta(x, y) < \delta \Rightarrow d(f'(x), f'(y)) < \varepsilon$ if

$x, y \in S \subset E$, $d(f'(x), f'(y)) = d(f'(x), f'(c)) < \varepsilon$

if $y = c$.

<u>Case 3</u>. $\bar{S}$ has at least two elements

Let $f'(x) = \lim_{n\to\infty} f(a_n)$, $f'(y) = \lim_{n\to\infty} f(b_n)$.

Let $\varepsilon > 0$, since $f'(x) = \lim_{n\to\infty} f(a_n)$

$\exists N_1 > 0$, s.t. $n > N_1 \Rightarrow d(f'(x), f(a_n)) < \frac{\varepsilon}{3}$

Let $d(x, a_n) = \frac{1}{n}$. this is equivalent to

$\exists \delta_1 = \frac{1}{N_1} > 0$, s.t. $d(x, a_n) < \delta_1 \Rightarrow d(f'(x), f(a_n)) < \frac{\varepsilon}{3}$,

Similarly, since $f'(y) = \lim_{n\to\infty} f(b_n)$

$\exists \delta_2 = \frac{1}{N_2} > 0$, s.t. $d(y, b_n) < \delta_2 \Rightarrow d(f'(y), f(b_n)) < \frac{\varepsilon}{3}$

Since f is uniformly continuous in S,

$\exists \delta_3 > 0$, s.t. $d(a_n, b_n) < \delta_3 \Rightarrow d(f(a_n), f(b_n)) < \frac{\varepsilon}{3}$

Now, select $\delta = \min\{\frac{\delta_1}{3}, \frac{\delta_2}{3}, \frac{\delta_3}{3}\}$, then $N = \lceil \frac{1}{\delta} \rceil$

$N > N_1$, $N > N_2$. $\frac{1}{N} < \frac{\delta_3}{3} \Rightarrow$

① $d(x, a_n) < \delta = \frac{1}{N} < \frac{1}{N_1} = \delta_1$

② $d(y, b_n) < \delta = \frac{1}{N} < \frac{1}{N_2} = \delta_2$

③ $d(x,y) < \delta \Rightarrow d(a_n, b_n) < d(a_n, x) + d(x,y) + d(y, b_n)$

$$< \frac{1}{N} + \delta + \frac{1}{N}$$

$$< \frac{\delta_3}{3} + \frac{\delta_3}{3} + \frac{\delta_3}{3}$$

$$< \delta_3$$

$$\Rightarrow d(f'(x), f'(y)) < d(f'(x), f(a_n)) + d(f(a_n), f(b_n)) + d(f(b_n), f'(y))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, f' is uniformly continuous in E.